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A Parametric Method for Solving Certain Nonconcave Maximization Problems*

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ABSTRACT

A maximization problem with linear inequality constraints and different kinds of nonconcave objective functions is considered. By means of parametric quadratic programming, the solution of the original problem is reduced to the determination of the absolute maximum of a continuous function of one variable on a bounded interval.

1. INTRODUCTION

There exist well-known methods for the maximization of a concave objective function in a convex domain. In the case of a nonconcave objective function, however, these methods may not even converge to a local maximum. Therefore, methods for solving nonconcave maximization problems are available only for a few special types of these problems, such as quadratic maximization [*1*] and linear fractional programming (see, e.g., [*2*]–[*5*]).

In this paper a more general type of nonconcave maximization problem is considered. The linear fractional programming problem is a special case of the problem considered here.

The basic idea of the method of solution is to reduce this maximization problem to the determination of the maximum of a single-valued function of one variable on a bounded interval. To this end an additional constraint depending on a parameter is introduced in such a way that the objective function is concave on the feasible domain for any fixed value of the parameter. Finally, a method is developed for determining the optimal solution of this concave parametric problem as an explicit function of the parameter. Introducing this optimal solution into the objective function we obtain a function of the parameter whose absolute maximum is the solution of the original maximization problem.

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2. Statement of the Problem

In the following we are concerned with a maximization problem whose feasible domain R is defined by

$$R = \{x : Ax \leqslant b\},$$

where A is a (m, n)-matrix and b is an m-column vector. We assume that R is bounded and does not contain the point $x = 0$.

Let C be a symmetric and nonsingular (n, n)-matrix. Suppose there exist an n-column vector d and a $(n - 1, n)$-matrix M with rank $n - 1$ such that

$$Md = 0, d'x > 0 \qquad \text{for any} \qquad x \in R$$

and

$$MCM'$$

is positive-semidefinite.[1]

The objective function $Q(x)$ of the considered maximization problem may be any one of the following three kinds:

$$Q(x) = c_1'x + (d'x)(c_2'x) + \frac{1}{d'x}\left(c_3'x - \frac{1}{2}x'Cx\right) \tag{2.1}$$

or

$$Q(x) = c_1'x + \frac{c_3'x}{dx} + (d'x)\left(c_2'x - \frac{1}{2}x'Cx\right) \tag{2.2}$$

or

$$Q(x) = c_1'x + (d'x)(c_2'x) + \frac{c_3'x}{d'x} - \frac{1}{2}x'Cx \tag{2.3}$$

where c_1, c_2, and c_3 are n-column vectors.

Of course, $Q(x)$ may also be any function which arises from (2.1) to (2.3) by specializing; for example,

$$Q(x) = c_1'x + c_3'x/d'x.$$

This example shows that linear fractional programming problems are special cases of the problems considered in this paper.

An investigation of the method developed in the following sections shows that this method can also be applied to problems containing $(d'x)^k$ instead of $d'x$, where k is an arbitrary real number.

[1] This assumption implies that the quadratic form $-x'Cx$ is concave in the hyperplane defined by $d'x = t$ where t is an arbitrary real number.

3. The Method of Solution

The basic idea of the method for solving the problems formulated in Section 2 is to introduce the additional constraint

$$d'x = t,$$

where t is a real parameter. This reduces the original maximization problem to a parametric maximum problem.

Corresponding to the objective functions (2.1) to (2.3) we have

$$\max \left\{ \left(c_1 + tc_2 + \frac{c_3}{t}\right)' x - \frac{1}{2t} x'Cx \,\middle|\, Ax \leqslant b, d'x = t \right\} \tag{3.1}$$

$$\max \left\{ \left(c_1 + tc_2 + \frac{c_3}{t}\right)' x - \frac{t}{2} x'Cx \,\middle|\, Ax \leqslant b, d'x = t \right\} \tag{3.2}$$

$$\max \left\{ \left(c_1 + tc_2 + \frac{c_3}{t}\right)' x - \frac{1}{2} x'Cx \,\middle|\, Ax \leqslant b, d'x = t \right\}. \tag{3.3}$$

To unify the problem we introduce a function $\omega(t)$ which may be one of the following four functions:

$$\omega_1(t) = t^{-1}, \qquad \omega_2(t) = t, \qquad \omega_3(t) \equiv 1, \qquad \omega_4(t) \equiv 0,$$

as the special problem under consideration requires. The case $\omega = \omega_4$ corresponds to a linear problem which will be considered in Section 5.

Using the above convention we obtain the following problem.

$$\max \left\{ \left(c_1 + tc_2 + \frac{c_3}{t}\right)' x - \frac{\omega(t)}{2} x'Cx \,\middle|\, Ax \leqslant b, d'x = t \right\}. \tag{3.4}$$

Because of the assumption about the quadratic form $-x'Cx$, the objective function of problem (3.4) is concave on the feasible domain for any fixed value of t.

Parametric quadratic maximum problems with strictly concave objective functions have been investigated [*6*]. In the following we outline a method which is a generalization of results obtained in [*6*] and which enables us to express the optimal solution of (3.4) as an explicit function of t.

Let $t = \bar{t}$ be a value of the parameter for which (3.4) has a feasible solution. Then we solve (3.4) for $t = \bar{t}$ by means of one of the known methods for quadratic maximization problems. Let x_j be the optimal solution. Using the Kuhn-Tucker conditions [*7*], which for any fixed t are necessary and sufficient for the optimal solution, we can express x_j as a continuous function $x_j(t)$ of t and determine the interval $t_j \leqslant \bar{t} \leqslant t_{j+1}$ for which $x_j(t)$ is optimal.

For $t > t_{j-1}$ some other subset of the constraints becomes active and we can again express the optimal solution as a continuous function $x_{j+1}(t)$ of the parameter. $x_{j+1}(t)$ remains optimal for a certain interval $t_{j+1} \leqslant t \leqslant t_{j+2}$.

It can be shown that, after a finite number of steps, a value t^0 is reached such that the feasible domain is empty for $t > t^0$. Similarly, decreasing t we obtain, after a finite number of steps, a value t_0 such that the feasible domain is empty for $t < t_0$.

Finally, introducing the optimal solutions $x_j(t)$; $j = 0, 1,..., k$; into the objective function $Q(x)$, we obtain a continuous function $\Phi(t)$ defined as follows:

$$\Phi(t) = \begin{array}{lll} Q[x_0(t)] & \text{for} & t_0 \leqslant t \leqslant t_1, \\ Q[x_1(t)] & \text{for} & t_1 \leqslant t \leqslant t_2, \\ \vdots & & \vdots \qquad \vdots \\ Q[x_k(t)] & \text{for} & t_k \leqslant t \leqslant t^0. \end{array}$$

The absolute maximum of $\Phi(t)$ on the interval $[t_0, t^0]$ can be determined by means of a search method.

Let $\Phi(t)$ take its absolute maximum at t^* and suppose $t^* \in [t_{j_0}, t_{j_0+1}]$. Then, denoting the feasible domain of problem (3.4) by $R(t)$, we have

$$\Phi(t^*) = Q[x_{j_0}(t^*)] = \max_{t\in[t_0,t^0]} \{ \max_{x\in R(t)} Q(x)\}.$$

Therefore, $x_{j_0}(t^*)$ is the optimal solution of the maximization problem under consideration.

We conclude this section by proving a lemma which will be needed in what follows.

LEMMA 1. *Let B be a nonsingular (n, n)-matrix; b, a and e n-vectors and α and t real numbers. Suppose that for $t \leqslant t_0$ $(t_0 > 0)$ there exists an $x_0 = x_0(t)$ satisfying the following conditions:*

(1) $\qquad Bx_0 = b + et;$

(2) $\qquad a'x_0 < \alpha \quad$ *for* $\quad t < t_0$;

(3) $\qquad a'x_0 = \alpha \quad$ *for* $\quad t = t_0$.

(a) *If* $(B'^{-1}a)_i \leqslant 0$, $i = 1,..., n$, *then the conditions*

(4) $\qquad Bx \leqslant b + et, \qquad a'x \leqslant \alpha$

are inconsistent for $t > t_0$.

(b) *If* $(B'^{-1}a)_j > 0$, *for at least one* j *then the system*

$$\bar{B}x = \bar{b} + \bar{e}t, \qquad b_jx < (b)_j + (e)_j t$$

is consistent for $t > t_0$, *where* b_j *denotes the* kth *row of* B *and* $\bar{B}x = \bar{b} + \bar{e}t$ *arises from* (1) *by exchanging the* jth *row for* (3).

Proof. (a) Let u be an n-vector and consider the transformation

$$x = B^{-1}b + B^{-1}et - B^{-1}u.$$

Then the conditions (4) are equivalent to

$$(*) \qquad u \geqslant 0 \quad \text{and} \quad -a'B^{-1}u \leqslant \alpha - a'B^{-1}b - a'B^{-1}et.$$

From (1)–(3) it follows that

$$\alpha - a'B^{-1}b - a'B^{-1}et_0 = 0 \quad \text{and} \quad a'B^{-1}e > 0.$$

Thus $(*)$ is inconsistent for $t > t_0$ if $(B'^{-1}a)_i \leqslant 0$ for each i.

(b) If $(B'^{-1}a)_j > 0$ then (4) is satisfied for $t > t_0$ with $(u)_i = 0$, $i \neq j$, $(u)_j = a'B^{-1}e(t - t_0)/(B'^{-1}a)_j > 0$. This completes the proof.

4. The Quadratic Case

In this section we consider the case where (3.4) has a quadratic objective function i.e., $\omega(t) \neq \omega_4(t)$.

Let $\bar{x}$ be an arbitrary point satisfying the conditions $Ax \leqslant b$ and choose $\bar{t} = d'\bar{x}$. For $t = \bar{t}$ (3.4) is a concave problem and can be solved by one of the known methods. Suppose $x_j = x_j(\bar{t})$ is the optimal solution of (3.4) for $t = \bar{t}$.

We partition A and b into A_1, A_2 and $\bar{b}_1$, $\bar{b}_2$, respectively, in such a way that

$$\begin{aligned} A_1x_j &= \bar{b}_1, \qquad A_2x_j < \bar{b}_2, \\ d'x_j &= 0 + \bar{t}. \end{aligned} \tag{4.1}$$

If we write (4.1) in the form

$$B_jx_j = b_j + e_jt, \qquad \bar{B}_jx_j < \bar{b}_j, \tag{4.2}$$

it follows from the Kuhn–Tucker theorem [7] that, for $t = \bar{t}$, $x_j = x_j(\bar{t})$ satisfies the following conditions:

$$\begin{aligned} &(\alpha) \qquad \omega(t)\, Cx_j + B_j' \binom{u_j}{\lambda_j} = c_1 + tc_2 + c_3/t, \\ &(\beta) \qquad B_jx_j = b_j + te_j, \\ &(\gamma) \qquad u_j \geqslant 0, \end{aligned} \tag{4.3}$$

where u_j is an n-vector and λ_j is a scalar.

If we assume that the rows of B_j are linearly independent (for the case of dependent rows of B_j see Section 6), we can obtain x_j as a function of t from (4.3).

By assumption C^{-1} exists and (4.3α) yields

$$x_j = C^{-1}\left[c_1 + tc_2 + \frac{c_3}{t} - B_j' \binom{u_j}{\lambda_j}\right] \frac{1}{\omega(t)}. \tag{4.4}$$

Introducing (4.4) into (4.3β) we obtain

$$-\frac{1}{\omega(t)} B_j C^{-1} B_j' \binom{u_j}{\lambda_j} = b_j + te_j - B_j C^{-1}\left[c_1 + tc_2 + \frac{c_3}{t}\right] \frac{1}{\omega(t)}.$$

From this expression we get

$$\binom{u_j}{\lambda_j} = h_j^1 + h_j^2 t + h_j^3 \frac{1}{t} + h_j^4 \omega(t) + h_j^5 t\omega(t), \tag{4.5}$$

where

$$M = [B_j C^{-1} B_j']^{-1}, \qquad h_j^1 = MB_j C^{-1} c_1, \qquad h_j^2 = MB_j C^{-1} c_2,$$
$$h_j^3 = MB_j C^{-1} c_3, \qquad h_j^4 = -Mb_j, \qquad h_j^5 = -Me_j.$$

If we introduce (4.5) into (4.4) and use the abbreviations

$$g_j^1 = -C^{-1}B_j'h_j^4, \qquad g_j^2 = -C^{-1}B_j'h_j^5, \qquad g_j^3 = C^{-1}c_1 - C^{-1}B_j h_j^1,$$
$$g_j^4 = C^{-1}c_2 - C^{-1}B_j'h_j^2, \qquad g_j^5 = C^{-1}c_3 - C^{-1}B_j'h_j^3,$$

we obtain

$$x_j(t) = g_j^1 + g_j^2 t + g_j^3 \frac{1}{\omega(t)} + g_j^4 \frac{t}{\omega(t)} + g_j^5 \frac{1}{t\omega(t)}. \tag{4.6}$$

By the Kuhn–Tucker theorem the vector x_j given by (4.6) is the optimal solution of (3.4) for all t for which

$$\begin{aligned} &(\alpha) \quad u_j(t) \geqslant 0 \qquad [u_j(t) \ \text{ given by } \ (4.5)], \\ &(\beta) \quad \bar{B}_j x_j \leqslant \bar{b}_j. \end{aligned} \tag{4.7}$$

If we choose $\omega(t) = \omega_1(t)$ or $\omega(t) = \omega_3(t)$ it follows immediately from (4.5) that the ith component of u_j can be written as follows:

$$(u_j)_i = t^{-1}[\beta_{i2}t^2 + \beta_{i1}t + \beta_{i0}] \tag{4.8}$$

where β_{i2}, β_{i1}, β_{i0} are real numbers.

If $\omega(t) = \omega_2(t)$ we have

$$(u_j)_i = t^{-2}[\gamma_{i3}t^3 + \gamma_{i2}t^2 + \gamma_{i1}t + \gamma_{i0}]. \tag{4.9}$$

$\gamma_{i\nu}$; $0 \leqslant \nu \leqslant 3$; are real numbers.

By evaluating the real roots of the polynomials in (4.8) [resp. (4.9)], we can determine the interval containing $\bar{t}$ for which $(u_j)_i \geqslant 0$.

Let I_j^1 be the intersection of those intervals determined for the components of u_j. Then $u_j(t) \geqslant 0$ for $t \in I_j^1$.

If we introduce (4.6) into (4.7β) and choose $\omega(t) = \omega_1(t)$ or $\omega(t) = \omega_3(t)$, it follows readily that each of the conditions (4.7β) can be written as follows:

$$1/\delta(t)[\bar{\beta}_{i2}t^2 + \bar{\beta}_{i1}t + \bar{\beta}_{i0}] \geqslant 0, \tag{4.10}$$

where $\bar{\beta}_{i2}$, $\bar{\beta}_{i1}$, $\bar{\beta}_{i0}$ are real numbers and

$$\delta(t) = \begin{matrix} 1 & \text{if} & \omega(t) = \omega_1(t), \\ t & \text{if} & \omega(t) = \omega_3(t). \end{matrix}$$

For $\omega(t) = \omega_2(t)$ each of the conditions (4.7β) can be written in the form

$$t^{-2}[\bar{\gamma}_{i3}t^3 + \bar{\gamma}_{i2}t^2 + \bar{\gamma}_{i1}t + \bar{\gamma}_{i0}] \geqslant 0. \tag{4.11}$$

Again by evaluating the real roots of the polynomials in (4.10), respectively (4.11), we can determine the interval containing $\bar{t}$ for which the ith of the conditions (4.7β) is satisfied. Denote the intersection of those intervals by I_j^2. Then the $x_j(t)$ given by (4.6) is the optimal solution of problem (3.4) for all t with

$$t \in I_j = I_j^1 \cap I_j^2.$$

We denote the lower and upper bound of the closed interval I_j with t_j and t_{j+1}, respectively.

Now we have to consider the two different cases in which t_{j+1} is determined by the fact that, for $t > t_{j+1}$, exactly one of the conditions (4.7α) or (4.7β) is violated.[2]

If for $t > t_{j+1}$ a component of u_j becomes negative, this means that, for $t = t_{j+1}$, the corresponding constraint is superfluous for the determination of $x_j(t)$. Therefore, we cancel in (4.3) the corresponding column and row.

If for $t > t_{j+1}$ one of the constraints (4.7β), which we denote by $a'x \leqslant \alpha$, is violated, this constraint becomes active for $t = t_{j+1}$. Therefore, we add this constraint to the system (4.3). First, we suppose that a' is linearly independent from the rows of B_j. Then we can apply the procedure described above to the new system (4.3) and obtain an $x_{j+1}(t)$ which is the optimal solution of (3.4) for any

$$t \in I_{j+1} = \{t : t_{j+1} \leqslant t \leqslant t_{j+2}\}.$$

Notice that $x_j(t_{j+1}) = x_{j+1}(t_{j+1})$.

[2] The case in which (for $t > t_{j+1}$) several of the conditions (4.7) are violated is considered in Section 6.

Now let a' be a linear combination of the rows of B_j which we assume to be a square matrix.[3] There are two cases to consider:

(1) *The vector* $B_j'^{-1}a$ *has at least one positive component.* Let $(B_j'^{-1}a)_i > 0$; $i = 1,..., n_1$; and

$$g = -\omega(t_{j+1})\, Cx_j + c_1 + t_{j+1}c_2 + c_3/t_{j+1} .$$

If

$$\frac{(B'^{-1}g)_{i_0}}{(B'^{-1}a)_{i_0}} \geqslant \frac{(B'^{-1}g)_i}{(B'^{-1}a)_i}, \qquad i = 1,..., n_1 , \tag{4.12}$$

we obtain the new system (4.3) by replacing the i_0th of Eqs. (4.3β) by $a'x = \alpha$ and applying the described procedure to this new system.

It follows from Lemma 1 that $x_{j+1}(t)$ is feasible for some interval $t_{j+1} \leqslant t \leqslant t_{j+2}$. Furthermore, (4.12) implies $u_{j+1}(t_{j+1}) \geqslant 0$.

(2) *The vector* $B_j'^{-1}a$ *has no positive component.* In this case Lemma 1 states that there are no feasible points for $t > t_{j+1}$.

It remains to show that, after a finite number of steps, we obtain a value t^0 such that no feasible point exists for $t > t^0$. But this is rather obvious since the problem (3.4) has only a finite number of constraints and any subset of constraints determines the optimal solution for at most a finite number of intervals I_j as it follows immediately from the way in which the I_j's are determined.

After having obtained t^0 we determine the optimal solution as a function of t for $t \leqslant \bar{t}$. Using the described method, after a finite number of steps, we obtain a value t_0 such that the feasible domain is empty for $t < t_0$.

Finally, introducing the optimal solutions $x_j(t)$ into $Q(x)$ we obtain the function (3.8) whose absolute maximum is the solution of the considered maximization problem.

5. The Linear Case

If we choose $\omega(t) = \omega_4(t) \equiv 0$ the objective function of problem (3.4) becomes linear. We can, therefore, suppose that the optimal solution is always an extreme point of the feasible domain. In this case the system (4.3) has the following form:

$$\begin{aligned} &(\alpha) \quad B_j' \begin{pmatrix} u_j \\ \lambda_j \end{pmatrix} = c_1 + tc_2 + \frac{c_3}{t}, \\ &(\beta) \quad B_j x_j = b_j + te_j , \end{aligned} \tag{5.1}$$

where B_j is a nonsingular square matrix.

[3] If B_j is not a square matrix, we write $B_jx_j = b_j + e_jt$ and $a'x_j = \alpha$ in the form $B_j^1x_j^1 + B_j^2x_j^2 = b_j + e_jt$ and $a_1'x_j^1 + a_2'x_j^2 = \alpha$, where B_j^1 is a nonsingular square matrix. It is easy to see that the above consideration applies to B_j^1 and a_1.

From (5.1) we obtain

$$\begin{pmatrix} u_j \\ \lambda_j \end{pmatrix} = B_j'^{-1}\left(c_1 + tc_2 + \frac{c_3}{t}\right),$$
$$x_j = B_j^{-1}b_j + B_j^{-1}e_j t. \tag{5.2}$$

Using the same method as in Section 4 we determine the interval I_j for which $x_j = x_j(t)$ is the optimal solution of the considered problem.

If for $t > t_{j+1}$ the ith component of u_j becomes negative, the corresponding condition is superfluous for $t = t_{j-1}$ and for $t > t_{j+1}$ an adjacent extreme point of the feasible domain is optimal.

To obtain this extreme point we decrease in (5.1) the ith component of b_j, which means that x_j moves in the intersection of the $(n - 1)$ remaining hyperplanes, until a new constraint becomes active. This new constraint replaces the ith row of (5.1β) and the ith column of (5.1α).

If for $t > t_{j+1}$ one of the constraints not contained in (5.1β) becomes active, we have the same situation as in Section 4 in the case of a square matrix B_j.

All remaining steps are the same as in the quadratic case considered in Section 4.

6. Degenerated Cases

In this section we have to investigate some degenerated cases which have been excluded in the foregoing sections.

First we consider the case that for $t > t_{j+1}$ the optimal solution $x_j(t)$ lies in more than one additional hyperplane. Let these hyperplanes be denoted by

$$a_i'x_j = \alpha_i, \qquad i = 1,..., k.$$

We consider the point $\bar{x} = x_j(t_{j+1} + \epsilon)$, $\epsilon > 0$, and choose the real numbers $\delta_1, ..., \delta_k \geqslant 0$ in such a way that

$$a_i'\bar{x} < \alpha_i + \delta_i, \qquad i = 1,..., k.$$

Now we consider δ_1 as the parameter and decrease δ_1, for $t = t_{j+1} + \epsilon$, using the method described for the parameter t until $\delta_1 = 0$. Then we decrease δ_2 and so on until we have $\delta_1 = ... \delta_k = 0$.

If $\epsilon > 0$ is sufficiently small the optimal solution obtained in this way is, obviously,

$$x_{j+1}(t) \qquad \text{at the point} \qquad t = t_{j+1} + \epsilon.$$

It may happen, that decreasing δ_i, $1 \leqslant i \leqslant k$, we arrive at a value δ_i^0 such that for $\delta_i < \delta_i^0$ no feasible point exists.

Then it follows from Lemma 1 that the feasible domain is empty for $t \geqslant t_{j+1} + \epsilon$ and the procedure is finished.

The cases that for $t > t_{j+1}$ several components of u_j becomes negative or that for $t > t_{j+1}$ some components of u_j become negative and at the same time new conditions become active may be reduced to the above case by removing the conditions, corresponding to vanishing components of u_j, from the system (4.3) or (5.1) and dealing with them as new hyperplanes in which $x_j(t)$ lies for $t = t_{j+1}$.

7. A Computational Aspect

The determination of u_j and x_j as functions of t involves the computing of $M = [B_j C^{-1} B_j']^{-1}$.

As it has been shown in Section 4 the matrix $B_j C^{-1} B_j'$ is changed at any nondegenerated step in such a way that in the matrix B_j a row is canceled or a new row is added. Therefore, we can use the method given in [8] to compute the new matrix M from the old one.

Suppose the last row b_k' of B_j is canceled and denote the remaining matrix by B_{j+1}. Then we have

$$B_j C^{-1} B_j' = \begin{pmatrix} B_{j+1} C^{-1} B_{j+1}', & B_{j+1} C^{-1} b_k \\ b_k' C^{-1} B_{j+1}', & b_k' C^{-1} b_k \end{pmatrix}.$$

Furthermore, partition M into

$$\begin{pmatrix} M_1, & M_2 \\ M_2', & M_4 \end{pmatrix}$$

such that M_4 is a (1, 1)-matrix.

According to [8] we have

$$(B_{j+1} C^{-1} B_{j+1}')^{-1} = M_1 - M_2 M_4 M_2'.$$

Now suppose a new row a_1' is added to B_j. We have

$$(B_{j+1} C^{-1} B_{j+1}') = \begin{pmatrix} B_j C^{-1} B_j', & B_j C^{-1} a_1 \\ a_1' C^{-1} B_j', & a_1' C^{-1} a_1 \end{pmatrix} = \begin{pmatrix} D_1, & D_2 \\ D_2', & D_4 \end{pmatrix}.$$

Let $\gamma = D_4 - D_2' D_1^{-1} D_2$, where D^{-1} is known.

It is easy to see that $\gamma = 0$ if and only if the vector (D_2', D_4) is a linear combination of the rows of the matrix (D_1, D_2). But this fact implies that a_1' is linear-dependent from the rows of B_j. In this case one of the rows of B_j has to be exchanged against a_1' according to Section 4.

If $\gamma \neq 0$ it follows from [8] that

$$(B_{j+1}C^{-1}B_{j+1})^{-1} = \begin{pmatrix} D_1^{-1} + \frac{1}{\gamma} D_1^{-1} D_2 D_2' D_1^{-1}, & -\frac{1}{\gamma} D_1^{-1} D_2 \\ -\frac{1}{\gamma} D_2' D_1^{-1} \quad , & \frac{1}{\gamma} \end{pmatrix}.$$

References

1. K. Ritter. *Z. Wahrscheinlichkeitstheorie* **4**, 340-351 (1966).
2. A. Charnes and W. W. Cooper. *Naval Res. Logistics Quart.* **9**, 181-186 (1962).
3. W. Dinkelbach. *Z. Wahrscheinlichkeitstheorie* **1**, 141-145 (1962).
4. W. S. Dorn. "Linear Fractional Programming." *IBM Research Report* RC-830 (1962).
5. H. C. Joksch. *Naval Res. Logistics Quart.* **11**, 197-204 (1964).
6. K. Ritter. *Unternehmensforsch.* **6**, 149-166 (1962).
7. H. W. Kuhn and A. W. Tucker. *Proc. Second Berkeley Symp. Math. Stat. Prob.* **1**, 481-492 (1951).
8. R. A. Frazer, W. J. Duncan, and A. R. Collar. "Elementary Matrices," Cambridge University Press (1950).